THE APPROXIMATE SUMMATION OF n TERMS OF ANY HARMONIC SERIES.

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For any series $U_1+U_2+\ldots\ldots$, we shall give notations to certain groups of successive terms; thus,

$$G_1=U_{m+1}+U_{m+2}+\ldots\ldots+U_{m+x_1},\quad G_2=U_{m+x_1+1}+\ldots\ldots+U_{m+x_1+x_2},\ \ldots\ldots$$

Then for $n>m$, we have

$$(1)\qquad E\equiv(G_1+\ldots\ldots+G_n)-(U_1+\ldots\ldots+U_n)=(G_1-U_1)+\ldots\ldots+(G_n-U_n),$$

$$(2)\qquad E=(U_{n+1}+U_{n+2}+\ldots\ldots+U_{m+x_1+\ldots+x_n})-(U_1+U_2+\ldots\ldots+U_m),$$

since the terms U_{m+1}, ..., U_n are common to $G_1+\ldots\ldots+G_n$ and $U_1+\ldots\ldots+U_n$.

Applying this method to the special harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots$, and taking $m=1$, $x=y=w=\ldots\ldots=z=3$, we have, by (1) and (2),

$$(3)\qquad E=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\ldots+\left(\frac{1}{3n-1}+\frac{1}{3n}+\frac{1}{3n+1}\right).$$

$$-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)=\frac{2}{2.3.4}+\frac{2}{5.6.7}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)},$$

$$(4)\qquad E=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n+1}-1.$$

$$(5)\qquad \text{Hence } \frac{1}{n+1}+\ldots+\frac{1}{3n+1}=1+\frac{2}{2.3.4}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)}.$$

But

$$(6)\qquad \frac{2}{2.3.4}+\frac{2}{5.6.7}+\ldots\infty=.098612\ldots\ldots$$

$$(7)\qquad \frac{2}{2.3.4}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)}=.098612\ldots-\frac{1}{3(3n+1)(3n+2)}+\delta,$$

$$\delta<\frac{2}{(3n+1)(3n+2)(3n+3)(3n+4)}.$$

Now any number which gives the remainder 1 on division by 3 may be taken as the last in a group of $2n+1$ terms. We may therefore arrange the series $1+\frac{1}{2}+\ldots\ldots$ in successive groups of $2n+1$ terms by throwing out 0, 1, or 2 terms for each group, as the case may be.

(8) Hence $1+\frac{1}{2}+\dots\dots+\frac{1}{n'}=G+D+U$,

where G is the number of groups in n' terms, U is the sum of the deleted terms, and D is the sum of the difference groups, $\frac{2}{2.3.4}+\dots\dots+\frac{2}{(3n-1)(3n)(3n+1)}$. If n is not taken smaller than 15,

(9) $$D=(.098612\dots\dots\dots)G-\frac{10^G-1}{27(10^{G-1})\ (3g+1)\ (3g+2)}-\delta,$$

$$\delta<\frac{1}{216(3g+1)(3g+2)},$$

where g is the n of the group of lowest denominator.

Since the formula applies to the series $\frac{1}{r}+\frac{1}{r+1}+\dots\dots\dots$, $r>15$, we apply the following table for the sum of the earlier terms.

TABLE GIVING SUM OF FIRST n TERMS OF SERIES $1+\frac{1}{2}+\dots\dots\dots$

n	*Sum*	n	*Sum*
2	1.5	23	3.734291
3	1.833333	24	3.775958
4	2.083333	25	3.815958
5	2.283333	26	3.854419
6	2.449999	27	3.891456
7	2.592857	28	3.927171
8	2.717857	29	3.961653
9	2.828968	30	3.994987
10	2.928968	31	4.027245
11	3.019877	32	4.058495
12	3.103210	33	4.088798
13	3.180133	34	4.110209
14	3.251562	35	4.138781
15	3.318228	36	4.166559
16	3.380728	37	4.193586
17	3.439552	38	4.219901
18	3.495108	39	4.245542
19	3.547739	40	4.270542
20	3.597739	41	4.294933
21	3.645338	42	4.318742
22	3.690813	43	4.341998

Example. Find the sum approximately of $1+\frac{1}{2}+\ldots\ldots+\frac{1}{10,000}$.

10000	*Denominators of terms in U.*	
3) 9999	0	0
3333	3333	3332
3331		
3) 3330		
1110	1110	1109
1108		
3) 1107		
369	369	368
367		
3) 366		
122	122	0
121		
3) 120; ——40		

$$B=4.270542$$
$$U=0.016088$$
$$(1.098612)5=5.493061$$
$$9.779692$$
Second term of (9) $=-0.000025$

$$\text{Sum}=9.77966\ \ldots\ldots\ldots-\delta,\ \delta<.000,000,313.$$

Any harmonic series $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots\ldots$ may be reduced to the form

$$\frac{1}{d}\left(\frac{1}{n+f}+\frac{1}{n+1+f}+\frac{1}{n+2+f}+\ldots\ldots\ldots\right)$$

where f is the fractional part of the result of dividing a by d, and n is the integral part of the result. Also,

$$\frac{1}{d}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots\ldots+\frac{1}{n'}\right)-\frac{1}{d}\left(\frac{1}{n+f}+\frac{1}{n+1+f}+\ldots\ldots+\frac{1}{n'+f}\right)$$

$$=\frac{1}{d}\left(\frac{f}{n(n+f)}+\frac{f}{(n+1)(n+1+f)}+\ldots\ldots+\frac{f}{n'(n'+f)}\right).$$

Since the value of $\frac{1}{d}\left(\frac{1}{n}+\ldots\ldots+\frac{1}{n'}\right)$ may be found according to methods already

given, and since the series $\frac{1}{d}\left(\frac{f}{n(n+f)}+\ldots\ldots\right)$ converges rapidly, we may approximately determine the sum of n terms of any harmonic series.

NOTE ON THE MAXIMUM INDICATOR OF CERTAIN ODD NUMBERS.

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If p, the least prime factor of N, is of the form $4l+1$, the maximum indicator of N is a multiple of 4.

In the MONTHLY, May, 1905, p. 107, I have shown that, for p a prime of the form $4l+1$, we have

$$\left(1.2.3\ldots\ldots\frac{p-1}{2}\right)^2 \equiv -1 \pmod{p}.$$

Hence, if $\left(1.2.3\ldots\ldots\frac{p-1}{2}\right)^{2n} \equiv 1 \pmod{N}$, n is even; and therefore the maximum indicator of N, being a multiple of the least value $2n$ satisfying the above congruence, is also a multiple of 4.

Corollary. *The equation $y^4=mx+1$ has at least one positive integral solution when the least factor of m is congruent to unity modulo 4.*

By Wilson's theorem, it is easily shown as above that

The maximum indicator of any odd number is even.

Corollary. *The equation $y^2=mx+1$ has at least one positive integral solution, when m is odd.*

If p and $2p-1$ are odd primes, the maximum indicator of $p(2p-1)$ is a multiple of 4.

As in the first congruence above we have

$$(1.2.3\ldots\ldots\overline{p-1})^2 \equiv -1 \pmod{2p-1}.$$

By Wilson's theorem,

$$(1.2.3\ldots\ldots\overline{p-1})^2 \equiv 1 \pmod{p}.$$

$$\text{Thus } (1.2.3\ldots\ldots\overline{p-1})^4 \equiv 1 \pmod{p.\overline{2p-1}}.$$

Now, $(1.2.3\ldots\ldots\overline{p-1})$ is not congruent to 1 modulo $p.\overline{2p-1}$. It is thus shown that one indicator, at least, is 4. Hence the maximum indicator is a multiple of 4.